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RENEWAL PROCESS IN QUEUING PROBLEM AND REPLACEMENT OF MACHINE

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ABSTRACT
The paper is studying the renewal process. There are many fields in which it's used for example: the renewable energy (solar energy, wind energy), maintenance and replacement decision models, queuing system and other field in electronics. In this study our objective is to find the probability density function related to a renewal at a given time. Also to drive the renewal equation, how to find the current lifetime and residual life time. We obtain that as expected value of a renewal process which is the renewal function or mean-value function of the renewal process. Also we use the renewal equation and renewal reward processes in the applications. The solutions of these applications show a real answers due to reality conditions within a given accuracy.

KEYWORDS: renewal process, forward recurrence times, backward recurrence times, renewal reward processes.

INTRODUCTION
We can say that a Poisson process is a counting process for which the times between successive events are independent and identically distributed exponential random variables. One possible generalization is to consider a counting process for which the times between successive events are independent and identically distributed with an arbitrary distribution. Such a counting process is called a renewal process.[1-2]

Renewal theory is the branch of probability theory that generalizes Poisson processes for arbitrary holding times. The Poisson process is a continuous-time Markov process on the positive integers (usually starting at zero) which has independent identically distributed holding times.[3-4]

Renewal process:
Let x_1 be the time at which the first event has occurred. Let x_n be the random variable which represents the time between $(n-1)^{th}$ and n^{th} events of the process N(t), where $n \geq 2$ and N(t) is the number of event occurring in the time (0, t].

When an event of N(t) occurs, it is said that a renewal or regeneration has taken place. N(t) represents the number of events that have occurred by time t. The process composed of the inter occurrence times of renewal event is called the renewal process [5].The counting process $\{N(t), t \geq 0\}$ is a renewal process if the sequence of non-negative random variables $\{x_n, n\geq 1\}$ is independent and identically distributed. Also we can define another process using the interarrival times x_n. Let $s_0 = 0$ and $s_n = x_1 + x_2 + ... + x_n$, $n \geq 1$. s_n represents the waiting time until the n^{th} event occurrence or renewal. The process$\{s_1, s_2, ..., s_n\}$ is called as renewal process.[6]
Sample evolution of a renewal process with holding times S_i and jump times J_n as shown in figure 1.

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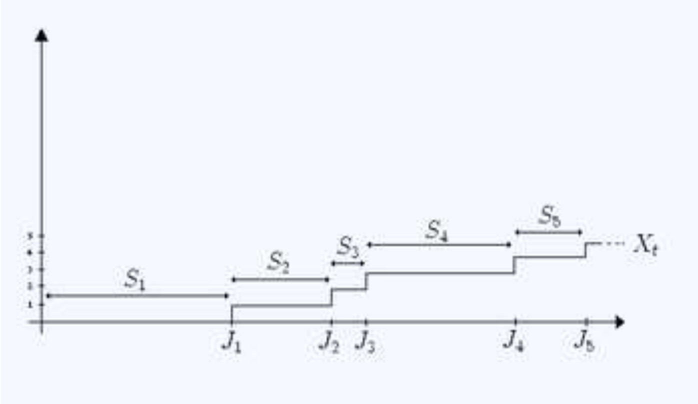

figure (1)

The distribution of N(t) can be obtained, at least in theory, by first noting the important relationship that the number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before or at time t.[7]

The study of the renewal process involves the study of the following:
1) Distribution of N(t).
2) Expected number of renewals E[N(t)] in time t.
3) The probability density function related to a renewal at a given time.
4) Time needed for specific number of events to occur.
5) Distributions of time since the last occurrence of a renewal event(backward recurrence time)and time till the next occurrence of a renewal event(forward recurrence time). The backward recurrence time is also called as age or current lifetime. The forward recurrence time is also called as excess life time or residual life time.

MATERIALS AND METHODS

To obtain the distribution of N(t) we use the important relationship that the number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before or at time t.

Distribution of N(t)

Let F denote the interarrival distribution and assume that F(0) = P{x_n =0}< 1. The number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before or at time t. [8]

so, N(t) $\geq n \;\Leftrightarrow\; s_n \;\leq t$

$$\therefore \;\; P[N(t) = n = P[N(t) \geq n] - P[N(t) \geq n+1]$$
$$= P[s_n \leq t] - P[s_{n+1} \leq t]$$

as $s_n = \sum_{i=1}^{n} x_i$ and all x_i's are i.i.d random variables with common distribution function F, s_n is distributed as n-fold convolution of F with itself, i.e.,F_n.

$$\therefore \;\; P[N(t) = n] = F_n(t) - F_{n+1}(t).$$

Renewal Function

If {N(t), t ≥ 0} is a renewal process, then the function U(t) = E[N(t)] is define for all t > 0 and it is called the renewal function or mean-value function of the renewal process.

$$E[n(t)] = \sum_{n=0}^{\infty} nP[N(t) = n]$$

= P[N(t)=1]+ 2P[N(t)=2] + 3P[N(t)=3]+...
= P[N(t)=1]+ P[N(t)=2] +P[N(t)=3 +...+
P[N(t)=2] +P[N(t)=3 +...+
P[N(t)=3 +...+
= P[(N(t)≥ 1] + P[(N(t)≥ 2] + P[(N(t)≥ 3] + ...

$$= \sum_{n=1}^{\infty} P[N(T) \geq n]$$
$$= \sum_{n=1}^{\infty} F_n\,(t).$$

Let the renewal function E[N(t)] denoted by U(t).

Renewal Density

The derivation of the renewal function is called as the renewal density. It denotes the probability density of a renewal at time t.

$$\frac{d}{dt}E[N(t)] = \frac{dU(T)}{dt} = \frac{d}{dt}\sum_{n=1}^{\infty} F_n(t)$$

$$= \sum_{n=1}^{\infty} \frac{d}{dt} F_n(t)$$

$$= \sum_{n=1}^{\infty} f_n(t) = u(t)$$

where u(t) is called the renewal density.

u(t)dt = P[a renewal will take place between (t, t+dt)]

and $f_n(t)$ is the interval density for the n^{th} renewal.[9-10]

Renewal Equations

A renewal equation is an integral equation satisfied by the renewal or mean-value function. This is derived by conditioning on the time of the first renewal. It can be solved some time to obtain renewal function.[11]

The Laplae transform of u(t) is:

$$L[u(t)] = \int_0^{\infty} e^{-st} u(t)dt$$

$$= \int_0^{\infty} e^{-st} \sum_{n=1}^{\infty} f_n(t)dt$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-st} f_n(t)dt$$

$$= \sum_{n=1}^{\infty} L[f_n(t)]$$

Let the distribution of the waiting time x_1 be $f_1(t)$ and the common distribution $x_n(n = 2, 3, 4, ...)$ be f(t).

so, $f_n(t) = f_1(t) * [f(t)]^{n-1}$, where * represents convolution operation.

$L[f_n(t)] = L[f_1(t)]\ [L(f(t))]^{n-1}$

$$L[u(t)] = \sum_{n=1}^{\infty} \mathrm{L}[\mathrm{f_n}(t)] = \sum_{n=1}^{\infty} L(f_1(t))[L(f(t))]^{n-1}$$

$$= L(f_1(t)) \sum_{n=1}^{\infty} [L(f(t))]^{n-1}$$

$$\frac{L[f_1(t)]}{1 - L[f(t)]}$$

Consider the renewal function U(t) and renewal density u(t).

$$L[U(t)] = \int_0^{\infty} e^{-st} U(t)dt$$

But, $\frac{d}{dt}U(t) = u(t)$

$$L[u(t)] = \int_0^{\infty} e^{-st} u(t)dt$$

so, $L[u(t)] = [e^{-st}U(t)]_0^{\infty} + s\int_0^{\infty} e^{-st}\, U(t)dt$

As U(0)=0,

L[u(t)] = s L[U(t)]

$$sL[U(t)] = \frac{L[f_1(t)]}{1 - L[f(t)]}$$

solving the above equation,

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$$L[U(t)] - L[U(t)]L[f(t)] = \frac{1}{s}L[f_1(t)]$$

We know that $L[f_1(t)] = \int_0^\infty e^{-st} f_1(t)dt$

If $F_1(t)$ is the probability distribution function corresponding to density function $f_1(t)$, then:

$$L[f_1(t)] = [e^{-st}F_1(t)]_0^\infty + s\int_0^\infty e^{-st}F_1(t)dt = sL[F_1(t)]$$

$$\frac{1}{s}L[f_1(t)] = L[F_1(t)]$$

then, $L[U(t) = L[U(t)]L[f(t)] + L[F_1(t)]$

Taking Inverse Laplace transform on both sides,

$$U(t) = F_1(t) + \int_0^t U(t-\tau)\,f(\tau)d\tau \qquad (*)$$

Differentiating both the sides of the above equation with respect to t,

$$u(t) = f_1(t) + \int_0^t u(t-\tau)\,f(\tau)d\tau \qquad (**)$$

The equations (*) and (**) are called renewal equations.

Asymptotic Renewal theorem

If N(t), t≥ 0 is a renewal process with $E[X_n]= \mu \; for\ all\ n,$ then

$$\frac{N(t)}{t} \rightarrow \frac{1}{\mu}\ as\ t \rightarrow \infty$$

Note: i) The number $\frac{1}{\mu}$ is called the rate of the renewal process.

ii) The above theorem is true even when the mean time between renewal μ, is infinite, in which case, $\frac{1}{\mu}$ is taken as 0.

Elementary Renewal Theorem

This theorem state that the expected average renewal rate converges to $\frac{1}{\mu}\ as\ t \rightarrow \infty$.

$$\frac{E[N(t)]}{t} \rightarrow \frac{1}{\mu}\ as\ t \rightarrow \infty$$

Relation between U(t) and $E[S_{N(t)+1}]$

Le t $S_{N(t)+1}$ represent the time of the first renewal after time t. We can derive a relationship between U(t), the mean number of renewals by time t and $E[S_{N(t)+1}]$, the expected time of the first renewal after t. [12]

It can be find that: U(t) = $\frac{E[SN(t)+1]}{\mu} - 1$

Then, $E[S_{N(t)+1}] = \mu[U(t) + 1]$

Various types of Renewal process

1) Let us assume that failure occurs at times $S_1, S_2, ..., S_n$ where $\{S_i\}$ are independent, non-negative continuous random variables. If $\{X_1, X_2, ...\}$are independent and identically distributed random variables with probability density function f(x), then the process is an ordinary renewal process.

2) The time from the origin to the first failure may have a different distribution $f_1(x)$ while $X_1, X_2, ...$ may have the same distribution f(x). Such a process is called a modified renewal process.

3) If in a modified renewal process the variable X_1 has a pdf $\frac{F(x)}{E(x_i)}, i = 2,3, ..$ then the process is called as equilibrium or stationary renewal process. An equilibrium renewal process can be considered as an ordinary renewal process where the system has been in use for a long time before it is first observed.[13]

Forward and Backward Recurrence Times

Let a renewal process be observed at time t. the time that has elapsed since the last renewal has taken place is called backward recurrence time. The random variable characterizing this time is also called as current life or age random variable. The backward recurrence time is denoted as $R^-(t)$.

$R^-(t) = t - S_{N(T)}$

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The time until the next renewal point from the present time is called forward recurrence time. The random variable characterizing this time is also called as excess or residual life time. The forward recurrence time is represented by $R^+(t)$.

$R^+(t) = S_{N(T)+1} - t$

The total life is the sum of current life and excess life time is:

$R(t) = R^-(t) + R^+(t)$

To find the density function of forward recurrence time. Let $r^+(x)$ represent the density of forward recurrence time.

$$\therefore p[x \leq R^+(t) \leq x + dx] = r^+(x)dx$$

Assume that a renewal takes place between v and v + dv prior to t, with probability u(v) dv, 0< v< t.

By renewal equation,

$$r^+(x) = f(t+x) + \int_0^t u(v)f(t+x-v)dv$$

Let y = t - v

$$r^+(x) = f(t+x) + \int_0^t u(t-y)f(x+y)dy$$

The first term of the above equation gives the probability of the first renewal at (t + x) and the second term gives the probability of a further renewal at (t + x) given that the immediate past renewal had taken place at v for $0 \leq v \leq t$.

by renewal theorem

$As\ t \to \infty,\ u(t) \to \frac{1}{\mu}, \forall t$

$$\lim_{t\to\infty} r^+ = \int_0^\infty \frac{1}{\mu} f(x+y)dy$$

Let x + y = w

$$\therefore \lim_{t\to\infty} r^+ = \int_x^\infty \frac{1}{\mu} f(w)dw = \frac{1-F(x)}{\mu}$$

Density function of Backward Recurrence time

Let $r^-(x)$ be the density function of backward recurrence time.[14-15]

The probability to have one renewal between (t -x) and (t - x + dx) is u(t - x)dx. The probability of having no renewal for further length of time x is 1 -F(x), so:

$r^-(x) = u(t - x) . (1 - F(x))$

$$\therefore \lim_{t\to\infty} r^- = \frac{1-F(x)}{\mu}$$

Central Limit Theorem for Renewal Processes

$$\lim_{t\to\infty} P\{\frac{N(t) - t/\mu}{tr^2/\mu^3} < x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$

In addition, as might be expected from the central limit theorem for renewal processes, it can be shown that Var(N(t))/t converges to σ2/μ3. That is,

$$\lim_{t\to\infty} \frac{var(N(T)}{t} = \frac{\sigma^2}{\mu^3}$$

Example : Two machines with continually process of an unending number of jobs. The time that it take to process a job on machine 1 is a gamma random variable with parameters n = 4, λ = 2, whereas the time that it takes to process a job on machine 2 is uniformly distributed between 0 and 4. Approximate the probability that together the two machines can process at least 90 jobs by time t =100. Solution: If we let Ni(t) denote the number of jobs that machine i can process by time t, then {N1(t),t ≥ 0} and {N2(t), t ≥ 0} are independent renewal processes. The interarrival distribution of the first renewal process is gamma with parameters n=4, λ =2, and thus has mean 2 and variance 1.[10] Correspondingly, the interarrival distribution of the second renewal process is uniform between 0 and 4, and thus has mean 2 and variance 16/12. Therefore, $N_1(100)$ is approximately normal with mean 50 and variance 100/8; and $N_2(100)$ is approximately normal with mean 50 and variance 100/6. Hence, $N_1(100)$ + $N_2(100)$ is approximately normal with mean 100 and variance 175/6. Thus, with denoting the standard normal distribution function, we have

$$P\{N_1(100)+N_2(100)>89.5\} = P\left\{\frac{N_1(100)+N_2(100)-100}{\sqrt{\frac{175}{6}}}\right\} \approx \frac{89.5-100}{\sqrt{\frac{175}{6}}} \approx 1-\varphi\left(\frac{-10.5}{\sqrt{\frac{175}{6}}}\right)$$

$$\approx \varphi\left(\frac{10.5}{\sqrt{\frac{175}{6}}}\right) \approx \varphi(1.944) \approx 0.974$$

Renewal Reward Processes

A large number of probability models are special cases of the following model. Consider a renewal process {N(t), t ≥ 0} having interarrival times X_n, n ≥ 1, and suppose that each time a renewal occurs we receive a reward. We denote by R_n the reward earned at the time of the nth renewal. We shall assume that the R_n,
n ≥ 1, are independent and identically distributed. However, we do allow for the possibility that R_n may (and usually will) depend on X_n, the length of the nth renewal interval. [16] If we let

$$R(t) = \sum_{n-1}^{N(t)} R_n$$

then R(t) represents the total reward earned by time t. Let $E[R]=E[R_n]$, $E[X]=E[X_n]$ Proposition :
If E[R] < ∞and E[X] < ∞, then:
(a) with probability 1, $\lim_{t\to\infty}\frac{R(t)}{t} = \frac{E[R]}{E[X]}$
(b) $\lim_{t\to\infty}\frac{E[R(t)]}{t} = \frac{E[R]}{E[X]}$

Application I (A Car Buying Model)

The lifetime of a car is a continuous random variable having a distribution Hand probability density h. If there is a policy that a new car should be bought as soon as the old one either breaks down or reaches the age of T years. Suppose that a new car costs C_1 dollars and also that an additional cost of C_2 dollars is incurred whenever the car breaks down. Under the assumption that a used car has no resale value, what is the car long-run average cost? If we say that a cycle is complete every time the car will be replaced, then it follows from the proposition (with costs replacing rewards) that the long-run average cost equals $\frac{\text{E[cost incurred during a cycle]}}{\text{E[length of a cycle]}}$. [17]

Let X be the lifetime of the car during an arbitrary cycle, then the cost incurred during that cycle will be given by:
C_1, if X > T
$C_1 + C_2$, if X ≤ T
so the expected cost incurred over a cycle is:
$C_1P\{X > T\}+(C_1 + C_2)P\{X \le T\}=C_1 + C_2H(T)$
Also, the length of the cycle is:
X, if X ≤ T
T, if X > T
and so the expected length of a cycle is:

$$\int_0^T xh(x)dx + \int_T^0 Th(x)dx = \int_0^T xh(x)dx + T[1-H(T)]$$

Therefore, the long-run average cost will be

$$\frac{C_1 + C_2H(T)}{\int_0^T xh(x)\,dx + T[1-H(T)}$$

suppose that the lifetime of a car (in years) is uniformly distributed over (0, 10), and suppose that C1 is 3 (thousand) dollars and C2 is 1 2 (thousand) dollars. What value of T minimizes the long-run average cost? If we uses the value T, T ≤10, the long-run average cost equals:

$$\frac{3+\frac{1}{2}(\frac{T}{10})}{\int_0^T\left(\frac{x}{10}\right)dx + T(1-\frac{T}{10})} = \frac{60+T}{20T-T^2} = g(T)$$

To find the minimum value:

$$g'(T) = \frac{(20T - T^2) - (60 + T)(20 - 2T)}{(20T - T^2)^2} = 0$$

$T^2 + 120T - 1200 = 0$

which yields the solutions T ≈9.25 and T ≈−129.25 Since T ≤10, it follows that the optimal policy for Mr. Brown would be to purchase a new car whenever his old car reaches the age of 9.25 years.

Application II (Dispatching a Train)

Suppose that customers arrive at a train depot in accordance with a renewal process having a mean interarrival time μ. Whenever there are N customers waiting in the depot, a train leaves. If the depot incurs a cost at the rate of nc dollars per unit time when ever there are n customers waiting, what is the average cost incurred by the depot?

If we say that a cycle is completed whenever a train leaves, then the preceding is a renewal reward process. The expected length of a cycle is the expected time required for N customers to arrive and, since the mean interarrival time is μ, this equals E[length of cycle]=Nμ

If we let T_n denote the time between the nth and (n+1)st arrival in a cycle, then the expected cost of a cycle may be expressed as:

E[cost of a cycle]=E[$cT_1 + 2cT_2 + \cdots + (N-1)cT_{N-1}$]

which, since E[T_n]=μ, equals:

$$c\mu \frac{N}{2}(N-1)$$

Hence, the average cost incurred by the depot is:

$c\mu \frac{N}{2N\mu}(N-1) = c\frac{(N-1)}{2}$. [18]

Suppose that each time a train leaves, the depot incurs a cost of six units. What value of N minimizes the depot's long-run average cost when c=2,μ =1?

In this case, we have that the average cost per unit time N is:

$$\frac{6 + c\mu N(N-1)}{N\mu} = N - 1 + \frac{6}{N}$$

By treating this as a continuous function of N and using the calculus, we obtain that the minimal value of N is N =√6≈2.45

DISCUSSION AND RESULTS

1) There are various types of renewal process: ordinary renewal process, modified renewal process, and equilibrium or stationary renewal process.

2) If u(t) is the renewal density, then u(t)dt = P[a renewal will take place between (t, t+dt)] and $f_n(t)$ is the interval density for the n^{th} renewal.

3) The following equation is actually representing a renewal equation

$$u(t) = f_1(t) + \int_0^t u(t-\tau) f(\tau) d\tau ,$$

and we can use it to find forward and backward recurrence times.

4) Both the density function of forward and backward recurrence time are the same.

5) In application I (A Car Buying Model) :

We find that the optimal decision policy for replacement of a car would be to purchase a new car whenever an old car reaches the age of 9.25 years which is very logical in most good condition of work and a suitable maintenance.

6) In application II (Dispatching a Train):

Hence, the optimal integral value of N is either 2 which yields a value 4, or 3 which also yields the value 4. Hence, either N = 2 or N = 3 minimizes the depot's average cost.

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